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# ON SOME DIOPHANTINE EQUATIONS INVOLVING BALANCING NUMBERS

EULOGE TCHAMMOU AND ALAIN TOGBÉ

**ABSTRACT.** In this paper, we find all the solutions of the Diophantine equation  $B_1^p + 2B_2^p + \cdots + kB_k^p = B_n^q$  in positive integer variables  $(k, n)$ , where  $B_i$  is the  $i^{\text{th}}$  balancing number if the exponents  $p, q$  are included in the set  $\{1, 2\}$ .

## 1. INTRODUCTION

In 1999, A. Behera, and G.K. Panda [2] studied balancing numbers  $n \in \mathbb{Z}^+$  as solutions of the Diophantine equation

$$(1.1) \quad 1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

for some positive integer  $r$ , in which case the number  $r$  is called a *balancer* or a *cobalancing number*. If  $n$  is a balancing number with balancer  $r$ , then  $\frac{n(n-1)}{2} = nr + \frac{r(r+1)}{2}$ . This means that

$$(1.2) \quad r = \frac{-(2n+1) + \sqrt{8n^2+1}}{2} \quad \text{and} \quad n = \frac{2r+1 + \sqrt{8r^2+8r+1}}{2}.$$

Let  $B_n$  denote the  $n^{\text{th}}$  balancing number and  $b_n$  the  $n^{\text{th}}$  cobalancing number. Then,

$$\begin{aligned} B_1 = 1, \quad B_2 = 6 \quad \text{and} \quad B_{n+1} = 6B_n - B_{n-1}, \quad \text{for } n \geq 2, \\ b_1 = 0, \quad b_2 = 2 \quad \text{and} \quad b_{n+1} = 6b_n - b_{n-1} + 2, \quad \text{for } n \geq 2. \end{aligned}$$

From (1.2), we see that  $B_n$  is a balancing number if and only if  $8B_n^2 + 1$  is a perfect square and  $b_n$  is a cobalancing number if and only if  $8b_n^2 + 8b_n + 1$  is a perfect square. The numbers

$$C_n = \sqrt{8B_n^2 + 1} \quad \text{and} \quad c_n = \sqrt{8b_n^2 + 8b_n + 1}$$

are then called the  $n^{\text{th}}$  Lucas-balancing number and the  $n^{\text{th}}$  Lucas-cobalancing number, respectively. P.K. Ray [10] derived some nice results on balancing numbers and Pell numbers which are given by

$$P_0 = 0, \quad P_1 = 1 \quad \text{and} \quad P_n = 2P_{n-1} + P_{n-2},$$

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for  $n \geq 2$ . More generally, for  $n \geq 0$ ,  $P_{-n} = (-1)^{n+1}P_n$  (extension of the sequence for negative subscripts). Since an integer  $x$  is a balancing number if and only if  $8x^2 + 1$  is a square, we set  $8x^2 + 1 = y^2$ , so that  $y^2 - 8x^2 = 1$ , for some integer  $y \neq 0$ , which is a Pell's equation. The fundamental solution is  $(x_1, y_1) = (1, 3)$ . So  $y_n + x_n\sqrt{8} = (3 + \sqrt{8})^n$  for  $n \geq 1$  and hence  $y_n - x_n\sqrt{8} = (3 - \sqrt{8})^n$ . Put  $\gamma = 3 + \sqrt{8}$  and  $\delta = 3 - \sqrt{8}$ . The Binet's formula for balancing numbers is

$$(1.3) \quad B_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}, \quad \text{for } n \geq 1.$$

Putting  $\alpha = 1 + \sqrt{2}$  and  $\beta = 1 - \sqrt{2}$ , the roots of the characteristic quadratic equation  $x^2 - 2x - 1 = 0$  of the Pell's sequence  $(P_n)_{n \geq 0}$ , the Binet's formula for  $P_n$  is

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}, \quad \text{for } n \in \mathbb{Z}.$$

This easily implies that the inequalities

$$(1.4) \quad \alpha^{n-2} \leq P_n \leq \alpha^{n-1}$$

hold, for  $n \geq 1$ . Since  $\alpha^2 = \gamma$  and  $\beta^2 = \delta$ , we easily get that  $B_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}$  for  $n \geq 1$ . Thus, there is a correspondence between balancing numbers and Pell numbers. More precisely, we have that  $B_n = \frac{P_{2n}}{2}$ . See [7], [8] and [9] for further details.

The Diophantine equation

$$\sum_{j=1}^k jF_j^p = F_n^q$$

has been studied in 2018 by G. Soydan, L. Németh, and L. Szalay [11], where  $F_i$  is the  $i^{th}$  Fibonacci number. They solved this equation for  $(p, q) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ . Further, they conjectured that the only non-trivial solutions are given by:

$$F_4^2 = 9 = F_1 + 2F_2 + 3F_3, \quad F_8 = 21 = F_1 + 2F_2 + 3F_3 + 4F_4, \quad F_4^3 = 27 = F_1^3 + 2F_2^3 + 3F_3^3.$$

Later in 2019, K. Gueth, F. Luca and L. Szalay [4] confirmed the conjecture, for  $\max\{p, q\} \leq 10$ . This result is again improved by Altassan and Luca [1] who proved that if such equation is satisfied, then  $\max\{k, n, p, q\} \leq 10^{2500}$ . The authors of this paper studied a similar equation where the Fibonacci sequence is replaced by the Pell sequence. See [12].

A question is what will happen if Fibonacci numbers are replaced by balancing numbers. Therefore, in this paper, we investigate the Diophantine equation

$$(1.5) \quad B_1^p + 2B_2^p + \cdots + kB_k^p = B_n^q,$$

in positive integers  $k$  and  $n$ , where  $p$  and  $q$  are fixed in  $\{1, 2\}$ . We consider  $B_1^p = 1 = B_1^q$  as a trivial solution to (1.5). The main results proved in this paper are described as follows.

**Theorem 1.** *The Diophantine equation*

$$(1.6) \quad B_1 + 2B_2 + \cdots + kB_k = B_n$$

has only the trivial solution  $(k, n) = (1, 1)$ .

**Theorem 2.** *The Diophantine equation*

$$(1.7) \quad B_1^2 + 2B_2^2 + \cdots + kB_k^2 = B_n^2$$

*possesses only the trivial solution  $(k, n) = (1, 1)$ .*

**Theorem 3.** *The Diophantine equation*

$$(1.8) \quad B_1 + 2B_2 + \cdots + kB_k = B_n^2$$

*possesses only the trivial solution  $(k, n) = (1, 1)$ .*

**Theorem 4.** *The Diophantine equation*

$$(1.9) \quad B_1^2 + 2B_2^2 + \cdots + kB_k^2 = B_n$$

*possesses only the trivial solution  $(k, n) = (1, 1)$ .*

We will prove our main results using modular arithmetic. We organize this paper as follows. In Section 2, we will recall some known properties and prove key lemmas. Our main results will be proved in Sections 3–6.

## 2. SOME USEFUL LEMMAS

In this section, we present some useful lemmas. Some of them are a few well-known results and we also prove some preliminary results. We start by recalling Euler's totient function, denoted  $\varphi$ , which is defined for each positive integer  $n$  by the number of integers  $k$  in the range  $1 \leq k \leq n$  such that  $\gcd(n, k) = 1$ . The following lemma is a well-known result. One can see Theorem 2.8 of [6].

**Lemma 1** (Euler's Totient Theorem). *Let  $n$  be a positive integer. For each non-zero integer  $a$  relatively prime to  $n$ ,*

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

The next lemma is a collection of well-known results. One can see for instance Proposition 2.1, Proposition 2.2, Proposition 2.3, and Proposition 2.6 in [3] and [5].

**Lemma 2.** *Let  $k$  and  $n$  be arbitrary positive integers.*

- (i)  $B_{k+n}B_{k-n} - B_k^2 = -B_n^2$  if  $k > n$  (Catalan's identity). In particular,  $B_{n-1}B_{n+1} - B_n^2 = -1$  (Cassini's identity) and  $B_{2n+1} = B_{n+1}^2 - B_n^2$ .
- (ii)  $\sum_{j=1}^k B_j = \frac{B_{k+1} - B_{k-1}}{4}$ .
- (iii)  $\gcd(B_k, B_n) = B_{\gcd(k, n)}$ . In particular,  $B_k$  and  $B_n$  are coprime if and only if  $k$  and  $n$  are coprime.
- (iv)  $P_{n-1}P_{n+1} - P_n^2 = (-1)^n$ .
- (v)  $P_{k+n} = P_kP_{n+1} + P_{k-1}P_n$ . In particular,  $P_{2n+1} = P_n^2 + P_{n+1}^2$ .

**Remark 5.** Properties (iv) and (v) hold for any integers  $k$  and  $n$ , using the formula of the extension to negative subscripts.

We will prove the following results.

**Lemma 3.** *Let  $k$  be a positive integer.*

- (i)  $6B_k B_{k-1} = B_k^2 + B_{k-1}^2 - 1$  and  $33B_k^2 - B_{k-1}^2 = B_{2k+1} - 2$  if  $k \geq 2$ .
- (ii)  $\sum_{j=1}^k B_j^2 = \frac{B_{2k+1} - 2k - 1}{32}$ .
- (iii)  $\sum_{j=1}^k j B_j = \frac{k(B_{k+1} - B_k) - B_k}{4}$ .
- (iv)  $\sum_{j=1}^k j B_j^2 = \frac{k B_{2k+1} - B_k^2 - k(k+1)}{32}$ .
- (v)  $B_{k+n} - B_{k+1-n} = (B_{k+1} - B_k)(P_{2n-1} + P_{2n-2})$ , for  $n \in \{1, 2, \dots, k\}$ . In particular,  $B_{k+n} \equiv B_{k+1-n} \pmod{(B_{k+1} - B_k)}$ .
- (vi)  $4B_{k+1-n}^2 - 4B_n^2 + P_{4n-1} = P_{2k-4n+3}(B_{k+1} - B_k)$ , for  $n \in \{1, 2, \dots, k\}$ . In particular,  $4B_{k+1-n}^2 \equiv 4B_n^2 - P_{4n-1} \pmod{(B_{k+1} - B_k)}$ .

**Proof.**

- The property (i) can be obtained easily, using the recurrence formula of  $(B_n)_{n \geq 1}$  and Lemma 2 (i).
- We prove property (ii). It is obvious to see that the property is true for  $k = 1$ . Assume that  $k \geq 2$ . We have

$$(2.1) \quad \sum_{j=1}^k B_j^2 = \sum_{j=1}^k \left( \frac{P_{2j}}{2} \right)^2 = \frac{1}{4} \sum_{j=1}^k P_{2j}^2.$$

Observe that

$$\begin{aligned} \sum_{j=1}^k P_{2j}^2 &= 4 + \sum_{j=2}^k (2P_{2j-1} + P_{2j-2})^2 = 4 + \sum_{j=2}^k (4P_{2j-1}^2 + P_{2j-2}^2 + 4P_{2j-1}P_{2j-2}) \\ &= 4 + \sum_{j=2}^k (4P_{2j-1}^2 - P_{2j-2}^2 + 2P_{2j-2}P_{2j}) = 4 + \sum_{j=2}^k (6P_{2j-1}^2 - P_{2j-2}^2 - 2) \\ &= 4 + \sum_{j=2}^k (6(6P_{2j-2}^2 - P_{2j-3}^2 + 2) - P_{2j-2}^2 - 2) \\ &= 4 + \sum_{j=2}^k (35P_{2j-2}^2 - 6P_{2j-3}^2 + 10) \\ &= 4 + \sum_{j=2}^k (34P_{2j-2}^2 + (P_{2j-2}^2 - 6P_{2j-3}^2 + 2) + 8) \\ &= 4 + \sum_{j=2}^k (34P_{2j-2}^2 - P_{2j-4}^2 + 8) \\ &= 33 \sum_{j=1}^k P_{2j}^2 - 33P_{2k}^2 + P_{2k-2}^2 + 8k - 4. \end{aligned}$$

In the above chain, we used Lemma 2 (iv) to get  $P_{2j-2}P_{2j} = P_{2j-1}^2 - 1$ . So

$$(2.2) \quad \sum_{j=1}^k P_{2j}^2 = \frac{33P_{2k}^2 - P_{2k-2}^2 - 8k + 4}{32}.$$

Now (2.1) and (2.2) imply that

$$\sum_{j=1}^k B_j^2 = \frac{33P_{2k}^2 - P_{2k-2}^2 - 8k + 4}{128} = \frac{33B_k^2 - B_{k-1}^2 - 2k + 1}{32} = \frac{B_{2k+1} - 2k - 1}{32},$$

where we used (i) to get  $33B_k^2 - B_{k-1}^2 = B_{2k+1} - 2$ . Then, property (ii) is proved.

• The next step is to prove (iii). We have

$$\begin{aligned} \sum_{j=1}^k jB_j &= 13 + \sum_{j=3}^k jB_j = 13 + \sum_{j=3}^k (6jB_{j-1} - jB_{j-2}) = 13 + 6 \sum_{j=3}^k jB_{j-1} - \sum_{j=3}^k jB_{j-2} \\ &= 13 + 6 \sum_{j=3}^k [(j-1)B_{j-1} + B_{j-1}] - \sum_{j=3}^k [(j-2)B_{j-2} + 2B_{j-2}] \\ &= 13 + 6 \sum_{j=2}^{k-1} jB_j + 6 \sum_{j=2}^{k-1} B_j - \sum_{j=1}^{k-2} jB_j - 2 \sum_{j=1}^{k-2} B_j \\ &= 1 + 5 \sum_{j=1}^k jB_j + 4 \sum_{j=1}^k B_j - (5k+4)B_k + (k+1)B_{k-1}. \end{aligned}$$

So, we get

$$\begin{aligned} \sum_{j=1}^k jB_j &= \frac{(5k+4)B_k - (k+1)B_{k-1} - 4 \sum_{j=1}^k B_j - 1}{4} \\ &= \frac{(5k+5)B_k - (k+1)B_{k-1} - B_{k+1}}{4} \\ &= \frac{(k+1)(B_{k+1} - B_k) - B_{k+1}}{4} = \frac{k(B_{k+1} - B_k) - B_k}{4}, \end{aligned}$$

where we used Lemma 2 (ii). Then property (iii) is proved.

• Now, we will take care of (iv). One can easily see that the property is true for  $k = 1$ . Assume that  $k \geq 2$ .

$$\begin{aligned} \sum_{j=1}^k jB_j^2 &= B_1^2 + 2B_2^2 + \sum_{j=3}^k j(6B_{j-1} - B_{j-2})^2 \\ &= 73 + \sum_{j=3}^k j(36B_{j-1}^2 - 12B_{j-1}B_{j-2} + B_{j-2}^2) \\ &= 73 + \sum_{j=3}^k j(34B_{j-1}^2 - B_{j-2}^2 + 2) = 73 + 34 \sum_{j=3}^k jB_{j-1}^2 - \sum_{j=3}^k jB_{j-2}^2 + 2 \sum_{j=3}^k j \end{aligned}$$

$$\begin{aligned}
&= 73 + 34 \sum_{j=3}^k [(j-1)B_{j-1}^2 + B_{j-1}^2] - \sum_{j=3}^k [(j-2)B_{j-2}^2 + 2B_{j-2}^2] + 2 \sum_{j=3}^k j \\
&= 67 + 34 \sum_{j=2}^{k-1} jB_j^2 + 34 \sum_{j=2}^{k-1} B_j^2 - \sum_{j=1}^{k-2} jB_j^2 - 2 \sum_{j=1}^{k-2} B_j^2 + 2 \sum_{j=1}^k j \\
&= 33 \sum_{j=1}^k jB_j^2 + 32 \sum_{j=1}^k B_j^2 - 33kB_k^2 + (k+1)B_{k-1}^2 - 32B_k^2 + k(k+1) - 1,
\end{aligned}$$

where we used Lemma 3 (i) to get that  $12B_{j-1}B_{j-2} = 2B_{j-2}^2 + 2B_{j-1}^2 - 2$ . Then,

$$\begin{aligned}
(2.3) \quad 32 \sum_{j=1}^k jB_j^2 &= 33kB_k^2 - (k+1)B_{k-1}^2 + 32B_k^2 - k(k+1) + 1 - 32 \sum_{j=1}^k B_j^2 \\
&= k(33B_k^2 - B_{k-1}^2) + 33B_k^2 - B_{k-1}^2 - B_k^2 - k(k+1) + 1 - B_{2k+1} + 2k + 1 \\
&= kB_{2k+1} - B_k^2 - k(k+1),
\end{aligned}$$

where we used (ii) and (i). Finally, (2.3) implies that

$$\sum_{j=1}^k jB_j^2 = \frac{kB_{2k+1} - B_k^2 - k(k+1)}{32},$$

as expected.

• Next, we will prove (v). Let  $n$  be an element of  $\{1, 2, \dots, k\}$ . We have

$$\begin{aligned}
(2.4) \quad (B_{k+1} - B_k)(P_{2n-1} + P_{2n-2}) &= \left( \frac{P_{2k+2} - P_{2k}}{2} \right) (P_{2n-1} + P_{2n-2}) \\
&= P_{2k+1}(P_{2n-1} + P_{2n-2}).
\end{aligned}$$

On the other hand, using Lemma 2 (v), we obtain

$$\begin{aligned}
(2.5) \quad P_{2n+2k} &= P_{2n}P_{2k+1} + P_{2n-1}P_{2k} \\
&= 2P_{2k+1}(P_{2n-1} + P_{2n-2}) - P_{2n-2}P_{2k+1} + P_{2n-1}P_{2k}.
\end{aligned}$$

Then, we get

$$(2.6) \quad P_{2k+1}(P_{2n-1} + P_{2n-2}) = \frac{1}{2}(P_{2n+2k} + P_{2n-2}P_{2k+1} - P_{2n-1}P_{2k}).$$

Equations (2.4) and (2.6) imply that

$$(2.7) \quad (B_{k+1} - B_k)(P_{2n-1} + P_{2n-2}) = \frac{1}{2}(P_{2n+2k} + P_{2n-2}P_{2k+1} - P_{2n-1}P_{2k}).$$

Using again Lemma 2 (v), we have that

$$\begin{aligned}
(2.8) \quad P_{2k+2-2n} &= P_{2-2n+2k} = P_{2-2n}P_{2k+1} + P_{1-2n}P_{2k} \\
&= -P_{2n-2}P_{2k+1} + P_{2n-1}P_{2k},
\end{aligned}$$

where we used formulas of the extension of the sequence of Pell numbers for negative subscripts. Equation (2.8) implies that

$$(2.9) \quad P_{2n-2}P_{2k+1} - P_{2n-1}P_{2k} = -P_{2k+2-2n}.$$

Now, equations (2.7) and (2.9) imply that

$$(B_{k+1} - B_k)(P_{2n-1} + P_{2n-2}) = \frac{1}{2}(P_{2n+2k} - P_{2k+2-2n}) = B_{k+n} - B_{k+1-n},$$

and property (v) is proved.

• Finally, we will deal with property (vi). Let  $n$  be an element of  $\{1, 2, \dots, k\}$ . Using the Binet's formula for  $(P_n)_{n \geq 0}$ , we obtain

$$(2.10) \quad \begin{aligned} P_{2k-4n+3}(B_{k+1} - B_k) &= P_{2k-4n+3} \left( \frac{P_{2k+2} - P_{2k}}{2} \right) = P_{2k-4n+3}P_{2k+1} \\ &= \frac{1}{8} (\alpha^{4k-4n+4} + \beta^{4k-4n+4} - \alpha^{2k-4n+3}\beta^{2k+1} - \alpha^{2k+1}\beta^{2k-4n+3}). \end{aligned}$$

Observe that

$$(2.11) \quad \begin{aligned} -\alpha^{2k-4n+3}\beta^{2k+1} - \alpha^{2k+1}\beta^{2k-4n+3} \\ = -\alpha^{2k+1}\beta^{2k+1}\alpha^2\alpha^{-4n} - \alpha^{2k+1}\beta^{2k+1}\beta^2\beta^{-4n} = \alpha^2\alpha^{-4n} + \beta^2\beta^{-4n} \\ = \alpha^{4n-2} + \beta^{4n-2}, \end{aligned}$$

where we used the fact that  $\alpha^{2k+1}\beta^{2k+1} = (-1)^{2k+1} = -1$ . Then, equations (2.10) and (2.11) imply that

$$(2.12) \quad P_{2k-4n+3}(B_{k+1} - B_k) = \frac{1}{8} (\alpha^{4k-4n+4} + \beta^{4k-4n+4} + \alpha^{4n-2} + \beta^{4n-2}).$$

On the other hand, again using the Binet's formula for  $(P_n)_{n \geq 0}$ , we get that

$$(2.13) \quad \begin{aligned} 4B_{k+1-n}^2 - 4B_n^2 + P_{4n-1} &= P_{2k+2-2n}^2 - P_{2n}^2 + P_{2n-1}^2 + P_{2n}^2 = P_{2k+2-2n}^2 + P_{2n-1}^2 \\ &= \frac{1}{8} (\alpha^{4k-4n+4} + \beta^{4k-4n+4} + \alpha^{4n-2} + \beta^{4n-2}), \end{aligned}$$

where we used Lemma 2 (v) to get that  $P_{4n-1} = P_{2n-1}^2 + P_{2n}^2$ . From equations (2.12) and (2.13), we conclude that

$$4B_{k+1-n}^2 - 4B_n^2 + P_{4n-1} = P_{2k-4n+3}(B_{k+1} - B_k),$$

as expected.  $\square$

**Lemma 4.** *Let  $(U_n)_{n \geq 1}$  be the sequence defined by  $U_n = B_{2n+1} - B_{2n-1}$  and  $k$  be a positive integer.*

- (i) *The sequence  $(U_n)_{n \geq 1}$  satisfies  $U_1 = 34$ ,  $U_2 = 1154$  and  $U_n = 34U_{n-1} - U_{n-2}$ , for  $n \geq 3$ .*
- (ii)  *$U_{k+n} - U_{k+1-n} = 32B_{2n-1}B_{2k+1}$ , for  $n \in \{1, 2, \dots, k\}$ . In particular,  $U_{k+n} \equiv U_{k+1-n} \pmod{B_{2k+1}}$ , for  $n \in \{1, 2, \dots, k\}$ .*



**Proof.** One can check easily (i). We prove (ii). Let  $n \in \{1, 2, \dots, k\}$ . We have

$$(2.14) \quad U_{k+n} - U_{k+1-n} = B_{2k+2n+1} - B_{2k+2n-1} - (B_{2k-2n+3} - B_{2k-2n+1}).$$

By, Lemma 3 (i),  $B_{2k+2n+1} = 33B_{k+n}^2 - B_{k+n-1}^2 + 2$  and by Lemma 2 (i),  $B_{2k+2n-1} = B_{k+n}^2 - B_{k+n-1}^2$ , so that  $B_{2k+2n+1} - B_{2k+2n-1} = 32B_{k+n}^2 + 2$ . Similarly,  $B_{2k-2n+3} - B_{2k-2n+1} = 32B_{k-n+1}^2 + 2$ . Then, equation (2.14) becomes

$$U_{k+n} - U_{k+1-n} = 32(B_{k+n}^2 - B_{k-n+1}^2) = 32B_{2n-1}B_{2k+1},$$

where we used Lemma 2 (i) to get that  $B_{k+n}^2 - B_{k-n+1}^2 = B_{2n-1}B_{2k+1}$ . Then,  $U_{k+n} - U_{k+1-n} = 32B_{2n-1}B_{2k+1}$ , as expected, completing the proof of (ii).  $\square$

**Lemma 5.** *Let  $k$  be a positive integer. Then, we have*

$$B_k^{\varphi(B_{2k+1})-1} \equiv B_k - B_{k+2} \pmod{B_{2k+1}},$$

where  $\varphi$  is Euler's totient function.

**Proof.** We have

$$(2.15) \quad B_k(B_k - B_{k+2}) = B_k^2 - B_k B_{k+2} = B_k^2 - B_{k+1}^2 + 1 \equiv 1 \pmod{B_{2k+1}},$$

since by Lemma 2 (i) to get that  $B_k B_{k+2} = B_{k+1}^2 - 1$  and  $B_k^2 - B_{k+1}^2 = -B_{2k+1}$ . Multiplying both sides of (2.15) by  $B_k^{\varphi(B_{2k+1})-1}$ , we get that

$$(2.16) \quad (B_k - B_{k+2})B_k^{\varphi(B_{2k+1})} \equiv B_k^{\varphi(B_{2k+1})-1} \pmod{B_{2k+1}}.$$

Since  $k$  and  $2k+1$  are coprime,  $B_k$  and  $B_{2k+1}$  are coprime (see Lemma 2 (iii)). Then, by Lemma 1,  $B_k^{\varphi(B_{2k+1})} \equiv 1 \pmod{B_{2k+1}}$ , so that (2.16) leads to

$$B_k^{\varphi(B_{2k+1})-1} \equiv B_k - B_{k+2} \pmod{B_{2k+1}}.$$

$\square$

### 3. PROOF OF THEOREM 1

For  $k = 1, 2, \dots, 5$ , one can easily find the solutions mentioned in the statement of Theorem 1. So, we assume from now that  $k \geq 6$ . Using Lemma 3 (iii), equation (1.6) leads to

$$k = \frac{4B_n + B_k}{B_{k+1} - B_k} \in \mathbb{N}.$$

This last equation implies that  $4B_n + B_k \equiv 0 \pmod{B_{k+1} - B_k}$ . So, we study the sequence  $(B_m)_{m \geq 1}$  modulo  $B_{k+1} - B_k$  if  $k$  is fixed. Note that we just indicate a suitable value congruent to  $B_m$  modulo  $B_{k+1} - B_k$ , not always the smallest non-negative remainders. Since  $B_{k+i} \equiv B_{k+1-i} \pmod{B_{k+1} - B_k}$ , for  $i \in \{1, 2, \dots, k\}$  (see Lemma 3 (v)), the period having length  $4k+2$  can be given by

$$\begin{array}{c} \overbrace{1, 6, 35, \dots, B_k}^k, \overbrace{B_k, B_{k-1}, B_{k-2}, \dots, 35, 6, 1}^k, \overbrace{0, -1, -6, -35, \dots, -B_k}^{k+1}, \\ \overbrace{-B_k, -B_{k-1}, -B_{k-2}, \dots, -35, -6, -1, 0}^{k+1}. \end{array}$$

So either  $B_n \equiv 0$  or  $\pm B_i \pmod{(B_{k+1} - B_k)}$ , for some  $i \in \{1, 2, \dots, k\}$ . Hence,

$$4B_n + B_k \equiv B_k \text{ or } \pm 4B_i + B_k \pmod{(B_{k+1} - B_k)}.$$

Assume that  $4B_n + B_k \equiv B_k \pmod{(B_{k+1} - B_k)}$ . We have

$$0 < B_k < B_{k+1} - B_k = 6B_k - B_{k-1} - B_k = 5B_k - B_{k-1},$$

so that  $B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$ . Thus  $4B_n + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$ .

Assume now that  $4B_n + B_k \equiv \pm 4B_i + B_k \pmod{(B_{k+1} - B_k)}$ ,  $i \in \{1, 2, \dots, k\}$ .

• If  $1 \leq i \leq k-1$ , then we get that

$$0 < -4B_{k-1} + B_k \leq \pm 4B_i + B_k \leq 4B_{k-1} + B_k < 5B_k - B_{k-1} = B_{k+1} - B_k.$$

So  $\pm 4B_i + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$ .

• If  $i = k$ , then

$$4B_i + B_k = 5B_k = B_{k+1} - B_k + B_{k-1} \equiv B_{k-1} \pmod{(B_{k+1} - B_k)}.$$

But, since  $k \geq 6$ , we have  $0 < B_{k-1} < 5B_k - B_{k-1} = B_{k+1} - B_k$ , so that

$$B_{k-1} \not\equiv 0 \pmod{(B_{k+1} - B_k)}.$$

Then,  $-4B_i + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$ . Similarly,  $-4B_i + B_k = -3B_k$ . Since

$$-5B_k + B_{k-1} = -B_{k+1} + B_k \equiv 0 \pmod{(B_{k+1} - B_k)},$$

we get that  $-3B_k \equiv 2B_k - B_{k-1} \pmod{(B_{k+1} - B_k)}$ . But  $2B_k - B_{k-1} \not\equiv 0 \pmod{(B_{k+1} - B_k)}$  since  $0 < 2B_k - B_{k-1} < B_{k+1} - B_k$ . Then,  $-4B_i + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$ .

In conclusion, we get that

$$4B_n + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)},$$

for  $k \geq 6$ . So equation (1.6) has no more solutions when  $k \geq 6$ . The proof of Theorem 1 is complete.

#### 4. PROOF OF THEOREM 2

When  $k = 1, 2, \dots, 5$ , one can easily find the solution given in Theorem 2. So, we assume from now that  $k \geq 6$ . Using Lemma 3 (iv), equation (1.7) implies that

$$(4.1) \quad k = \frac{32B_n^2 + B_k^2 + k(k+1)}{B_{2k+1}}.$$

Observe that  $32B_n^2 = 33B_n^2 - B_{n-1}^2 - (B_n^2 - B_{n-1}^2) = B_{2n+1} - 2 - B_{2n-1}$  (see Lemma 3 (i) and Lemma 2 (i)), so that equation (4.1) becomes

$$(4.2) \quad k = \frac{U_n + B_k^2 + k(k+1) - 2}{B_{2k+1}} \in \mathbb{N},$$

where  $(U_m)_{m \geq 1}$  is the sequence defined by  $U_m = B_{2m+1} - B_{2m-1}$ . By Lemma 4 (i),  $U_1 = 34$ ,  $U_2 = 1154$  and  $U_m = 34U_{m-1} - U_{m-2}$ , for  $m \geq 3$ . Equation (4.2) implies that  $U_n + B_k^2 + k(k+1) - 2 \equiv 0 \pmod{B_{2k+1}}$ . So, we study the sequence  $(U_m)_{m \geq 1}$  modulo  $B_{2k+1}$ , if  $k$  is fixed. Since  $U_{k+i} \equiv U_{k+1-i} \pmod{B_{2k+1}}$ ,

for  $i \in \{1, 2, \dots, k\}$  (see Lemma 4 (ii)), the period having length  $2k + 1$  can be given by

$$\overbrace{34, 1154, \dots, U_{k-1}, U_k}^k \overbrace{U_k, U_{k-1}, \dots, 1154, 34, 2}^{k+1}.$$

So either  $U_n \equiv 2$  or  $U_j \pmod{B_{2k+1}}$ , for some  $j \in \{1, 2, \dots, k\}$ . Hence, we have

$$U_n + B_k^2 + k(k+1) - 2 \equiv B_k^2 + k(k+1) \text{ or } U_j + B_k^2 + k(k+1) - 2 \pmod{B_{2k+1}}.$$

Assume that  $U_n + B_k^2 + k(k+1) - 2 \equiv B_k^2 + k(k+1) \pmod{B_{2k+1}}$ . Using Lemma 3 (i), we have

$$\begin{aligned} B_{2k+1} &= 33B_k^2 - B_{k-1}^2 + 2 > 32B_k^2 = B_k^2 + 31\frac{P_{2k}^2}{4} \\ &> B_k^2 + 7P_{2k}^2 > B_k^2 + k(k+1), \end{aligned}$$

where we used the fact that  $7P_{2k}^2 \geq 7\alpha^{4(k-1)} > k(k+1)$ , for  $k \geq 1$ . Hence,  $0 < B_k^2 + k(k+1) < B_{2k+1}$ , so that  $B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}$ . Then, we get

$$U_n + B_k^2 + k(k+1) - 2 \not\equiv 0 \pmod{B_{2k+1}}.$$

Assume now that

$$U_n + B_k^2 + k(k+1) - 2 \equiv U_j + B_k^2 + k(k+1) - 2 \pmod{B_{2k+1}},$$

for some  $j \in \{1, 2, \dots, k\}$ .

- If  $j = k$ , then we get that

$$\begin{aligned} U_k + B_k^2 + k(k+1) - 2 &= B_{2k+1} - B_{2k-1} + B_k^2 + k(k+1) - 2 \\ &= B_{2k+1} + B_{k-1}^2 + k(k+1) - 2 \\ (4.3) \quad &\equiv B_{k-1}^2 + k(k+1) - 2 \pmod{B_{2k+1}}, \end{aligned}$$

where, we used Lemma 2 (i) to get that  $B_{2k-1} = B_k^2 - B_{k-1}^2$ . Using the previous case, we have that

$$0 < B_{k-1}^2 + k(k+1) - 2 < B_k^2 + k(k+1) < B_{2k+1},$$

so that

$$(4.4) \quad B_{k-1}^2 + k(k+1) - 2 \not\equiv 0 \pmod{B_{2k+1}}.$$

Congruences (4.3) and (4.4) imply that  $U_j + B_k^2 + k(k+1) - 2 \not\equiv 0 \pmod{B_{2k+1}}$ ,

- If  $1 \leq j \leq k-1$ , then we get that

$$\begin{aligned} U_j + B_k^2 + k(k+1) - 2 &\leq U_{k-1} + B_k^2 + k(k+1) - 2 \\ &= B_{2k-1} - B_{2k-3} + B_k^2 + k(k+1) - 2 \\ &= B_{2k-1} + B_{k-2}^2 + B_{2k-1} + k(k+1) - 2 \\ (4.5) \quad &= 2B_{2k-1} + k(k+1) + B_{k-2}^2 - 2, \end{aligned}$$

where, we used Lemma 2 (i) to get that  $B_{2k-3} = B_{k-1}^2 - B_{k-2}^2$  and  $B_k^2 - B_{k-1}^2 = B_{2k-1}$ . On the other hand, we have

$$\begin{aligned}
 & B_{2k+1} - 2B_{2k-1} - k(k+1) - B_{k-2}^2 + 2 \\
 &= B_{k+1}^2 - B_k^2 - 2(B_k^2 - B_{k-1}^2) - k(k+1) - B_{k-2}^2 + 2 \\
 (4.6) \quad &= 36B_k^2 - 12B_k B_{k-1} + B_{k-1}^2 - 3B_k^2 + 2B_{k-1}^2 - B_{k-2}^2 - k(k+1) + 2 \\
 &= 31B_k^2 + B_{k-1}^2 - B_{k-2}^2 - k(k+1) + 4 \\
 &> 31B_k^2 - k(k+1) + 4 = \frac{31}{4}P_{2k}^2 - k(k+1) + 4 > 0.
 \end{aligned}$$

In the above chain, we used Lemma 3 (i) to get that  $12B_k B_{k-1} = 2B_k^2 + 2B_{k-1}^2 - 2$ , as well as the fact that  $\frac{31}{4}P_{2k}^2 > 7\alpha^{4(k-1)} > k(k+1) - 4$ , for  $k \geq 1$ . Inequality (4.6) implies that

$$(4.7) \quad 2B_{2k-1} + k(k+1) + B_{k-2}^2 - 2 < B_{2k+1}.$$

Now, inequalities (4.5) and (4.7) imply that  $U_j + B_k^2 + k(k+1) - 2 < B_{2k+1}$ . Finally, we have  $0 < U_j + B_k^2 + k(k+1) - 2 < B_{2k+1}$ , so that

$$U_j + B_k^2 + k(k+1) - 2 \not\equiv 0 \pmod{B_{2k+1}}.$$

In conclusion, we have

$$U_j + B_k^2 + k(k+1) - 2 \not\equiv 0 \pmod{B_{2k+1}},$$

for  $j \in \{1, 2, \dots, k\}$ . Therefore, equation (1.7) has no more solutions for  $k \geq 6$ . This completes the proof of Theorem 2.

## 5. PROOF OF THEOREM 3

The proof of this theorem is similar to the proof of Theorem 1. For  $k = 1, 2, \dots, 5$ , one can easily find the solution mentioned in the statement of Theorem 3. So, we assume from now that  $k \geq 6$ . By Lemma 3 (iii), equation (1.8) implies that

$$k = \frac{4B_n^2 + B_k}{B_{k+1} - B_k} \in \mathbb{N}.$$

This last equation implies that  $4B_n^2 + B_k \equiv 0 \pmod{(B_{k+1} - B_k)}$ . So, we study here the sequence  $(B_m^2)_{m \geq 1}$  modulo  $B_{k+1} - B_k$ , if  $k$  is fixed. The period having length  $2k+1$  can be deduced from the range

$$\overbrace{1^2, 6^2, 35^2, \dots, B_k^2}^k, \overbrace{B_k^2, B_{k-1}^2, B_{k-2}^2, \dots, 35^2, 6^2, 1^2, 0}^{k+1}.$$

So we have  $B_n^2 \equiv 0$  or  $B_i^2 \pmod{(B_{k+1} - B_k)}$ , for some  $i \in \{1, 2, \dots, k\}$ . Thus

$$4B_n^2 + B_k \equiv B_k \text{ or } 4B_i^2 + B_k \pmod{(B_{k+1} - B_k)}.$$

- Assume that  $4B_n^2 + B_k \equiv B_k \pmod{(B_{k+1} - B_k)}$ . We have  $0 < B_k < B_{k+1} - B_k$ , so that  $B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$ . So  $4B_n^2 + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$ .

- Assume that  $4B_n^2 + B_k \equiv 4B_i^2 + B_k \pmod{(B_{k+1} - B_k)}$ , for some  $i \in \{1, 2, \dots, k\}$ .

We put  $m = \lfloor \frac{k}{2} \rfloor$ . Suppose that  $1 \leq i \leq m$ . Then, we have

$$(5.1) \quad 4B_i^2 + B_k = P_{2i}^2 + B_k \leq P_{2m}^2 + B_k \leq P_k^2 + B_k.$$

On the other hand, we get

$$(5.2) \quad \begin{aligned} B_{k+1} - B_k &= 4B_k - B_{k-1} + B_k = \frac{4P_{2k} - P_{2k-2}}{2} + B_k \\ &= \frac{8P_{2k-1} + 3P_{2k-2}}{2} + B_k = \frac{8P_{k-1}^2 + 8P_k^2 + 3P_{2k-2}}{2} + B_k \\ &= 4P_{k-1}^2 + 4P_k^2 + 3\frac{P_{2k-2}}{2} + B_k > P_k^2 + B_k. \end{aligned}$$

From (5.1) and (5.2), one can see that  $4B_i^2 + B_k < B_{k+1} - B_k$ . Thus we obtain

$$0 < 4B_i^2 + B_k < B_{k+1} - B_k,$$

so that  $4B_i^2 + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$ , for  $i \in \{1, 2, \dots, m\}$ .

It remains to prove that for  $i \in \{1, 2, \dots, m+1\}$ ,

$$4B_{k+1-i}^2 + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}.$$

By Lemma 3 (vi), we have  $4B_{k+1-i}^2 + B_k \equiv 4B_i^2 + B_k - P_{4i-1} \pmod{(B_{k+1} - B_k)}$ , for  $i \in \{1, 2, \dots, m+1\}$ . So, it suffices to prove that for  $i \in \{1, 2, \dots, m+1\}$ ,

$$4B_i^2 + B_k - P_{4i-1} \not\equiv 0 \pmod{(B_{k+1} - B_k)}.$$

Let  $i$  be an element of the set  $\{1, 2, \dots, m+1\}$ . From the previous argument, we get  $4B_i^2 + B_k < B_{k+1} - B_k$ . Then, we deduce that

$$(5.3) \quad 4B_i^2 + B_k - P_{4i-1} < B_{k+1} - B_k.$$

We will prove that  $4B_i^2 + B_k - P_{4i-1} > 0$ . We have

$$(5.4) \quad \begin{aligned} 4B_i^2 + B_k - P_{4i-1} &= P_{2i}^2 + \frac{P_{2k}}{2} - P_{2i-1}^2 - P_{2i}^2 = \frac{2P_{2k-1} + P_{2k-2}}{2} - P_{2i-1}^2 \\ &\geq \frac{2P_{k-1}^2 + 2P_k^2 + P_{2k-2}}{2} - P_{2m+1}^2 \\ &\geq P_{k-1}^2 + P_k^2 - P_{k+1}^2 + \frac{P_{2k-2}}{2} > 0. \end{aligned}$$

Now, from (5.3) and (5.4), we have  $0 < 4B_i^2 + B_k - P_{4i-1} < B_{k+1} - B_k$ , so that  $4B_i^2 + B_k - P_{4i-1} \not\equiv 0 \pmod{(B_{k+1} - B_k)}$ . Hence, we obtain

$$4B_{k+1-i}^2 + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)},$$

for  $i \in \{1, 2, \dots, m+1\}$ . This completes the proof of Theorem 3.

## 6. PROOF OF THEOREM 4

For  $k = 1, 2, \dots, 5$ , one can easily find the solution mentioned in the statement of Theorem 4. So, we assume that  $k \geq 6$ . By Lemma 2 (i), one can see that

$B_{2k}^2 = B_{2k-1}B_{2k+1} + 1 \equiv 1 \pmod{B_{2k+1}}$ . Using Lemma 3 (iv), equation (1.9) implies that

$$(6.1) \quad k = \frac{32B_n + B_k^2 + k(k+1)}{B_{2k+1}}.$$

Equation (6.1) implies particularly that  $32B_n + B_k^2 + k(k+1) \equiv 0 \pmod{B_{2k+1}}$ . So, here we study the sequence  $(B_m)_{m \geq 1}$  modulo  $B_{2k+1}$  if  $k$  is fixed. The period can be deduced from the range

$$\overbrace{1, 6, 35, \dots, B_{2k}, 0}^{2k+1} \overbrace{-B_{2k}, -6B_{2k}, -35B_{2k}, \dots, -B_{2k-1}B_{2k}, -B_{2k}^2, 0}^{2k+1}$$

of length  $4k+2$  since  $B_{2k}^2 \equiv 1 \pmod{B_{2k+1}}$ . So either  $B_n \equiv 0 \pmod{B_{2k+1}}$  or  $B_n \equiv B_j$  or  $-B_jB_{2k} \pmod{B_{2k+1}}$ , for some  $j \in \{1, 2, \dots, 2k\}$ . Hence, we have

$$32B_n + B_k^2 + k(k+1) \equiv B_k^2 + k(k+1) \pmod{B_{2k+1}}$$

or

$$32B_i + B_k^2 + k(k+1) \equiv 32B_j + B_k^2 + k(k+1) \pmod{B_{2k+1}}$$

or

$$32B_i + B_k^2 + k(k+1) \equiv -32B_jB_{2k} + B_k^2 + k(k+1) \pmod{B_{2k+1}},$$

for some  $j \in \{1, 2, \dots, 2k\}$ . Therefore, we will distinguish three cases.

**Case 1:**  $32B_n + B_k^2 + k(k+1) \equiv B_k^2 + k(k+1) \pmod{B_{2k+1}}$ . Using Lemma 3 (i), we have

$$B_{2k+1} = 33B_k^2 - B_{k-1}^2 + 2 > 32B_k^2 = B_k^2 + 31\frac{P_{2k}^2}{4} > B_k^2 + k(k+1),$$

since  $31\frac{P_{2k}^2}{4} > 7P_{2k}^2 > 7\alpha^{4(k-1)} > k(k+1)$ . Hence, we obtain,  $0 < B_k^2 + k(k+1) < B_{2k+1}$ , so that  $B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}$ . Then, we deduce that

$$32B_n + B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}.$$

**Case 2:**  $32B_n + B_k^2 + k(k+1) \equiv 32B_j + B_k^2 + k(k+1) \pmod{B_{2k+1}}$ , for some  $j \in \{1, 2, \dots, 2k\}$ .

• If  $j = 2k$ , then we get that

$$(6.2) \quad \begin{aligned} 32B_{2k} + B_k^2 + k(k+1) &= 5B_{2k+1} + 2B_{2k} + 5B_{2k-1} + B_k^2 + k(k+1) \\ &\equiv 2B_{2k} + 5B_{2k-1} + B_k^2 + k(k+1) \pmod{B_{2k+1}}. \end{aligned}$$

We will check that

$$(6.3) \quad 2B_{2k} + 5B_{2k-1} + B_k^2 + k(k+1) < B_{2k+1}.$$

Indeed, one can see that

$$5B_{2k-1} = B_{2k} + B_{2k-2} - B_{2k-1} < B_{2k},$$

so that

$$(6.4) \quad 2B_{2k} + 5B_{2k-1} + B_k^2 + k(k+1) < 3B_{2k} + B_k^2 + k(k+1).$$

On the other hand, using  $B_{2k} = \frac{B_{2k+1} + B_{2k-1}}{6}$  and Lemma 2 (i), we get that

$$\begin{aligned}
 (6.5) \quad & B_{2k+1} - 3B_{2k} - B_k^2 - k(k+1) \\
 &= 16B_k^2 + B_{k-1}^2 - 6B_k B_{k-1} - k(k+1) \\
 &= 15B_k^2 + 1 - k(k+1) = 15\frac{P_{2k}^2}{4} + 1 - k(k+1) > 0,
 \end{aligned}$$

where we used Lemma 3 (i) to get that  $6B_k B_{k-1} = B_k^2 + B_{k-1}^2 - 1$  and the fact that  $15\frac{P_{2k}^2}{4} > 3\alpha^{4(k-1)} > k(k+1)$ , for  $k \geq 1$ . Inequality (6.5) implies that

$$(6.6) \quad 3B_{2k} + B_k^2 + k(k+1) < B_{2k+1}.$$

From (6.4) and (6.6), we get  $2B_{2k} + 5B_{2k-1} + B_k^2 + k(k+1) < B_{2k+1}$ . So inequality (6.3) is proved and finally

$$(6.7) \quad 0 < 2B_{2k} + 5B_{2k-1} + B_k^2 + k(k+1) < B_{2k+1}.$$

Now, (6.2) and (6.7) imply that

$$32B_j + B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}.$$

• If  $1 \leq j \leq 2k-1$ , then we have

$$(6.8) \quad 32B_j + B_k^2 + k(k+1) \leq 32B_{2k-1} + B_k^2 + k(k+1).$$

Furthermore, using Lemma 2 (i) and the recurrence formula, we get

$$\begin{aligned}
 (6.9) \quad & B_{2k+1} - 32B_{2k-1} - B_k^2 - k(k+1) \\
 &= 2B_k^2 + 33B_{k-1}^2 - 12B_k B_{k-1} - k(k+1) \\
 &= 31B_{k-1}^2 + 2 - k(k+1) > 0,
 \end{aligned}$$

where we used again Lemma 3 (i), as well as the fact that  $31B_{k-1}^2 > 7P_{2k-2}^2 > 7\alpha^{4(k-2)} > k(k+1)$ , for  $k \geq 1$ . Inequality (6.9) implies that

$$(6.10) \quad 32B_{2k-1} + B_k^2 + k(k+1) < B_{2k+1}.$$

Now, (6.8) and (6.10) imply that  $32B_j + B_k^2 + k(k+1) < B_{2k+1}$ . Finally, we have

$$0 < 32B_j + B_k^2 + k(k+1) < B_{2k+1}, \text{ so that } 32B_j + B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}.$$

In all subcases, we have  $32B_j + B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}$ . So, we obtain

$$32B_n + B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}.$$

**Case 3:**  $32B_n + B_k^2 + k(k+1) \equiv -32B_j B_{2k} + B_k^2 + k(k+1) \pmod{B_{2k+1}}$ , for some  $j \in \{1, 2, \dots, 2k\}$ . We will prove that  $-32B_j B_{2k} + B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}$ . Assume that  $-32B_j B_{2k} + B_k^2 + k(k+1) \equiv 0 \pmod{B_{2k+1}}$  in order to get a contradiction. Then, one can see that

$$B_k^2 + k(k+1) \equiv 32B_j B_{2k} \pmod{B_{2k+1}}.$$

Since  $B_{2k}^{\varphi(B_{2k+1})} \equiv 1 \pmod{B_{2k+1}}$ , multiplying both sides by  $B_{2k}^{\varphi(B_{2k+1})-1}$ , we get

$$(6.11) \quad [B_k^2 + k(k+1)] B_{2k}^{\varphi(B_{2k+1})-1} \equiv 32B_j \pmod{B_{2k+1}}.$$

By Lemma 5,  $B_{2k}^{\varphi(B_{2k+1})-1} \equiv B_k - B_{k+2} \pmod{B_{2k+1}}$ . Then, (6.11) implies that

$$[B_k^2 + k(k+1)](B_k - B_{k+2}) \equiv 32B_j \pmod{B_{2k+1}},$$

i.e.

$$32B_j + [B_k^2 + k(k+1)](B_{k+2} - B_k) \equiv 0 \pmod{B_{2k+1}}.$$

This leads to

$$(6.12) \quad 32B_j + [B_k^2 + k(k+1)](B_{k+2} - B_k) - B_k B_{2k+1} \equiv 0 \pmod{B_{2k+1}},$$

since  $B_k B_{2k+1} \equiv 0 \pmod{B_{2k+1}}$ . Observe that

$$\begin{aligned} & [B_k^2 + k(k+1)](B_{k+2} - B_k) - B_k B_{2k+1} \\ &= [B_k^2 + k(k+1)](6B_{k+1} - 2B_k) - B_k(B_{k+1}^2 - B_k^2) \\ (6.13) \quad &= B_k B_{k+1}(6B_k - B_{k+1}) - B_k^3 + 6k(k+1)B_{k+1} - 2k(k+1)B_k \\ &= B_k B_{k+1} B_{k-1} - B_k^3 + 6k(k+1)B_{k+1} - 2k(k+1)B_k \\ &= 6k(k+1)B_{k+1} - [2k(k+1) + 1]B_k, \end{aligned}$$

where we used Lemma 2 (i) to get that  $B_{2k+1} = B_{k+1}^2 - B_k^2$ ,  $B_{k+1}B_{k-1} = B_k^2 - 1$ . Then, (6.12) and (6.13) imply that

$$(6.14) \quad 32B_j + 6k(k+1)B_{k+1} - [2k(k+1) + 1]B_k \equiv 0 \pmod{B_{2k+1}},$$

• Suppose that  $1 \leq j \leq 2k-2$ . We will prove that

$$0 < 32B_j + 6k(k+1)B_{k+1} - [2k(k+1) + 1]B_k < B_{2k+1}.$$

We have

$$(6.15) \quad \begin{aligned} & 32B_j + 6k(k+1)B_{k+1} - [2k(k+1) + 1]B_k \\ & \geq 32 + 6k(k+1)B_{k+1} - [2k(k+1) + 1]B_k > 0. \end{aligned}$$

On the other hand, since  $j \leq 2k-2$ , we have

$$\begin{aligned} & B_{2k+1} - 32B_j - 6k(k+1)B_{k+1} + [2k(k+1) + 1]B_k \\ & \geq B_{2k+1} - 32B_{2k-2} - 6k(k+1)B_{k+1} + [2k(k+1) + 1]B_k \\ &= B_{2k+1} - 32 \frac{B_{2k-1} + B_{2k-3}}{6} - 6k(k+1)B_{k+1} + [2k(k+1) + 1]B_k \\ &= B_{k+1}^2 - B_k^2 - \frac{16}{3}(B_k^2 - B_{k-1}^2 + B_{k-1}^2 - B_{k-2}^2) - 6k(k+1)B_{k+1} \\ & \quad + [2k(k+1) + 1]B_k \\ (6.16) \quad & \geq B_{k+1}^2 - B_k^2 - 6(B_k^2 - B_{k-2}^2) - 6k(k+1)B_{k+1} + [2k(k+1) + 1]B_k \\ &= B_k[36B_k - 6B_{k-1} - 36k(k+1)] - B_{k-1}[6B_k - B_{k-1} - 6k(k+1)] \\ & \quad - B_k[7B_k - 2k(k+1) - 1] + 6B_{k-2}^2 \\ & > B_k[29B_k - 6B_{k-1} - 34k(k+1) + 1] \\ & \quad - B_{k-1}[6B_k - B_{k-1} - 6k(k+1)]. \end{aligned}$$



Observe that

$$(6.17) \quad \begin{aligned} & 29B_k - 6B_{k-1} - 34k(k+1) + 1 - 6B_k + B_{k-1} + 6k(k+1) \\ &= 23B_k - 5B_{k-1} - 28k(k+1) + 1 > 18B_k - 28k(k+1) + 1 > 0, \end{aligned}$$

since  $18B_k = 9P_{2k} \geq 9\alpha^{2k-2} > 28k(k+1)$ , for  $k \geq 4$ . Now, (6.17) implies that

$$29B_k - 6B_{k-1} - 34k(k+1) + 1 > 6B_k - B_{k-1} - 6k(k+1).$$

Then, one obtains

$$(6.18) \quad B_k [29B_k - 6B_{k-1} - 34k(k+1) + 1] - B_{k-1} [6B_k - B_{k-1} - 6k(k+1)] > 0.$$

Inequalities (6.16) and (6.18) finally imply that

$$(6.19) \quad 32B_j + 6k(k+1)B_{k+1} - [2k(k+1) + 1] B_k < B_{2k+1}.$$

From (6.15) and (6.19), we deduce that

$$0 < 32B_j + 6k(k+1)B_{k+1} - [2k(k+1) + 1] B_k < B_{2k+1},$$

which contradicts (6.14). Then,  $j \in \{2k-1, 2k\}$ .

• Suppose that  $j = 2k-1$ . Then, (6.14) becomes

$$(6.20) \quad 32B_{2k-1} + 6k(k+1)B_{k+1} - [2k(k+1) + 1] B_k \equiv 0 \pmod{B_{2k+1}}.$$

Using the recurrence formula, we check that  $32B_{2k-1} = B_{2k+1} + B_{2k-3} - 2B_{2k-1}$ .

Then, (6.20) becomes

$$(6.21) \quad B_{2k+1} + B_{2k-3} - 2B_{2k-1} + 6k(k+1)B_{k+1} - [2k(k+1) + 1] B_k \equiv 0 \pmod{B_{2k+1}}.$$

We will prove that

$$0 < B_{2k+1} + B_{2k-3} - 2B_{2k-1} + 6k(k+1)B_{k+1} - [2k(k+1) + 1] B_k < B_{2k+1}.$$

It is obvious that

$$(6.22) \quad B_{2k+1} + B_{2k-3} - 2B_{2k-1} + 6k(k+1)B_{k+1} - [2k(k+1) + 1] B_k > 0.$$

Furthermore, one sees that

$$\begin{aligned} & B_{2k+1} - B_{2k+1} - B_{2k-3} + 2B_{2k-1} - 6k(k+1)B_{k+1} + [2k(k+1) + 1] B_k \\ &= 2B_k^2 - 3B_{k-1}^2 + B_{k-2}^2 - 36k(k+1)B_k + 6k(k+1)B_{k-1} + [2k(k+1) + 1] B_k \\ &> B_k^2 + B_{k-2}^2 - 36k(k+1)B_k + 6k(k+1)B_{k-1} + [2k(k+1) + 1] B_k \\ &= B_k [B_k - 34k(k+1) + 1] + B_{k-2}^2 + 6k(k+1)B_{k-1} > 0, \end{aligned}$$

where we used Lemma 2 (i), the fact that  $B_k^2 - 3B_{k-1}^2 > 0$  and at the end the fact that  $B_k = \frac{P_{2k}}{2} \geq \frac{1}{2}\alpha^{2k-2} > 34k(k+1) - 1$ , for  $k \geq 6$ , which is the case for us. The above inequality implies that

$$(6.23) \quad B_{2k+1} + B_{2k-3} - 2B_{2k-1} + 6k(k+1)B_{k+1} - [2k(k+1) + 1] B_k < B_{2k+1}.$$

From inequalities (6.22) and (6.23), we have

$$0 < B_{2k+1} + B_{2k-3} - 2B_{2k-1} + 6k(k+1)B_{k+1} - [2k(k+1) + 1] B_k < B_{2k+1},$$

which contradicts (6.21).

• Suppose that  $j = 2k$ . Then, (6.14) becomes

$$(6.24) \quad 32B_{2k} + 6k(k+1)B_{k+1} - [2k(k+1) + 1] B_k \equiv 0 \pmod{B_{2k+1}}.$$

Observe that

$$32B_{2k} = 5B_{2k+1} + 5B_{2k-1} + 2B_{2k} \equiv 5B_{2k-1} + 2B_{2k} \pmod{B_{2k+1}}.$$

Then, (6.24) implies

$$(6.25) \quad 5B_{2k-1} + 2B_{2k} + 6k(k+1)B_{k+1} - [2k(k+1) + 1] B_k \equiv 0 \pmod{B_{2k+1}}.$$

We prove that  $0 < 5B_{2k-1} + 2B_{2k} + 6k(k+1)B_{k+1} - [2k(k+1) + 1] B_k < B_{2k+1}$ . It is obvious that

$$(6.26) \quad 5B_{2k-1} + 2B_{2k} + 6k(k+1)B_{k+1} - [2k(k+1) + 1] B_k > 0.$$

On the other hand, using Lemma 2 (i), we get that

$$\begin{aligned} & B_{2k+1} - 5B_{2k-1} - 2B_{2k} - 6k(k+1)B_{k+1} + [2k(k+1) + 1] B_k \\ &= B_{2k+1} - 5B_{2k-1} - 2 \frac{B_{2k+1} + B_{2k-1}}{6} \\ (6.27) \quad &= \frac{1}{3} [72B_k^2 - 24B_k B_{k-1} + 2B_{k-1}^2 - 18B_k^2 + 16B_{k-1}^2] \\ &\quad + \frac{1}{3} [-108k(k+1)B_k + 18k(k+1)B_{k-1} + [6k(k+1) + 3] B_k] \\ &= \frac{1}{3} [B_k [50B_k - 102k(k+1) + 3] + B_{k-1} [14B_{k-1} + 18k(k+1)]] , \end{aligned}$$

where we used Lemma 3 (i) to get  $24B_k B_{k-1} = 4B_k^2 + 4B_{k-1}^2 - 4$ . Moreover,

$$\begin{aligned} & B_k [50B_k - 102k(k+1) + 3] + B_{k-1} [14B_{k-1} + 18k(k+1)] \\ (6.28) \quad &> B_k [50B_k - 102k(k+1) + 3] > 0, \end{aligned}$$

since  $50B_k = 25P_{2k} \geq 25\alpha^{2k-2} > 102k(k+1)$ , for  $k \geq 4$ . Now, (6.27) and (6.28) imply that  $B_{2k+1} - 5B_{2k-1} - 2B_{2k} - 6k(k+1)B_{k+1} + [2k(k+1) + 1] B_k > 0$ , i.e.

$$(6.29) \quad 5B_{2k-1} + 2B_{2k} + 6k(k+1)B_{k+1} - [2k(k+1) + 1] B_k < B_{2k+1}.$$

From inequalities (6.26) and (6.29), we deduce that

$$0 < 5B_{2k-1} + 2B_{2k} + 6k(k+1)B_{k+1} - [2k(k+1) + 1] B_k < B_{2k+1},$$

which contradicts (6.25). Then, the proof of Theorem 4 is complete.

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